Deriving Bisimulation Cogruences for Reactive Systems

A review of Leifer and Milner paper

Alessio Ferrarini (2076777)
11th of July 2023

Università degli Studi di Padova

Labels

Are the labels that we used in CCS reflecting how a process interacts?

Our reaction relation $\xrightarrow{a}$ was indexed by the actions, we redefine the transition as being indexed by the contexts that permit such action.

$$a \xrightarrow{F} a' \Leftrightarrow F[a] \rightarrow a'$$

Example:

Instead of $\bar{x}.\, A \mid B \xrightarrow{x} A$ we would write $\bar{x}.\, A \mid B \xrightarrow{x.0 \mid \cdot} A$

The issue

But this does not really capture the the fact that a process requires such context to react.

For the process $\bar{x}.\ A$ we could write $\bar{x}.\ A \xrightarrow{x.B+x.C} A$ and according to our definition of $\xrightarrow{F}$ this would still be correct.

We can have infinite contexts that would trigger a reaction but that don't encode any behaviuor.

Reactive system as a category

Definition Reactive system

A reactive system is a triple $(\mathbb{C}, \mathbb{G}, \text{React})$ where

- $\mathbb{C}$ is a category
- $0 \in |\mathbb{C}|$
- $\text{React} \subseteq \bigcup_m \mathbb{C}(0, m)^2$ is the set of reaction rules.
- $\mathbb{D} \subseteq \mathbb{C}$ made of reactive contexts and composition reflecting.

We use the 0 object to identify the arrows $0 \to m$ that are the agents.

Clearly the issue of "complex" labels is caused by out definiton of $\rightarrow$.

$$a \xrightarrow{F} a' \Leftrightarrow F[a] \rightarrow a'$$

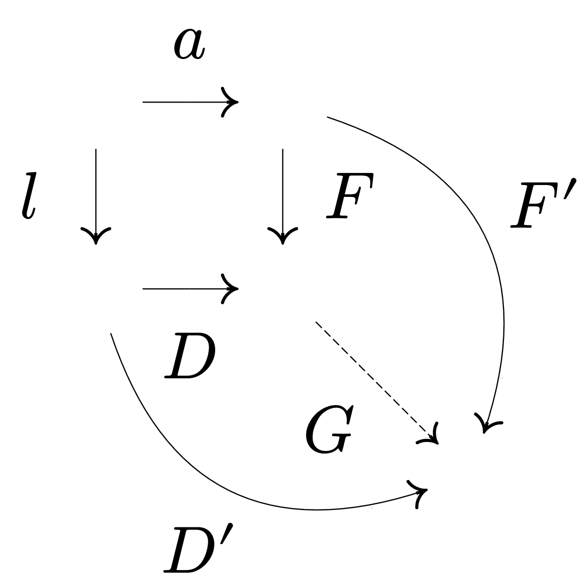
$$\Leftrightarrow \exists (l, r) \in \text{React } \exists D \quad F[a] = D[l] \text{ and } a' = D[r]$$

$$\begin{array}{ccc} & a & \\ & \longrightarrow & \\ l \downarrow & & \downarrow F \\ & \longrightarrow & \\ & D & \end{array}$$

But nothing forces F and G to be the "smallest" context making the diagram commute.

The fix

We would like something like:



For any other contexts F' and D' satisfying the same condition as F and D there exists an unique G such that $G \cdot F = F'$ and $G \cdot D = D'$.

If we think back to context we are looking to find the context F such that any other context F' triggering the same reaction can be factorized as

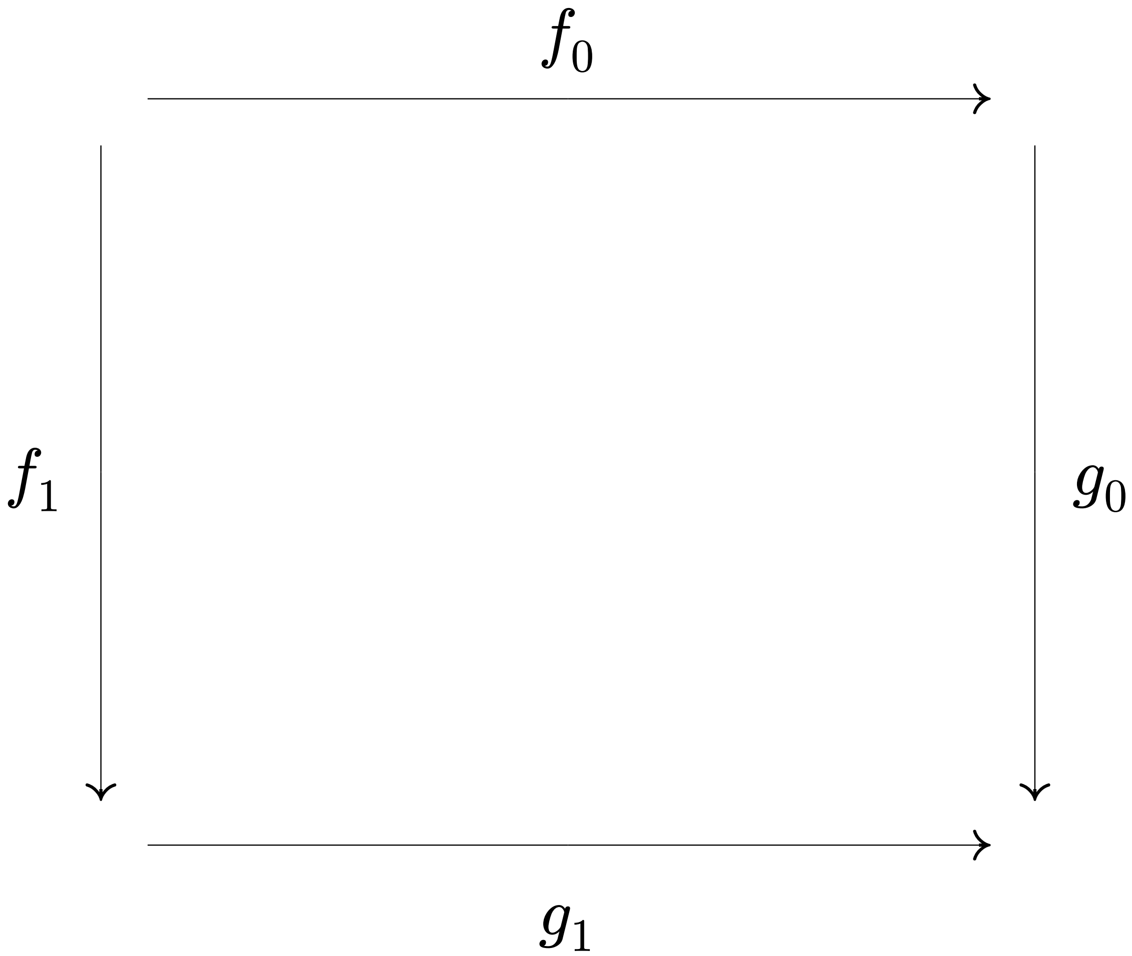
$$G \cdot F = F'$$

where in G we have captured all the useless complexity that wasn't really needed in F' to trigger the reaction.

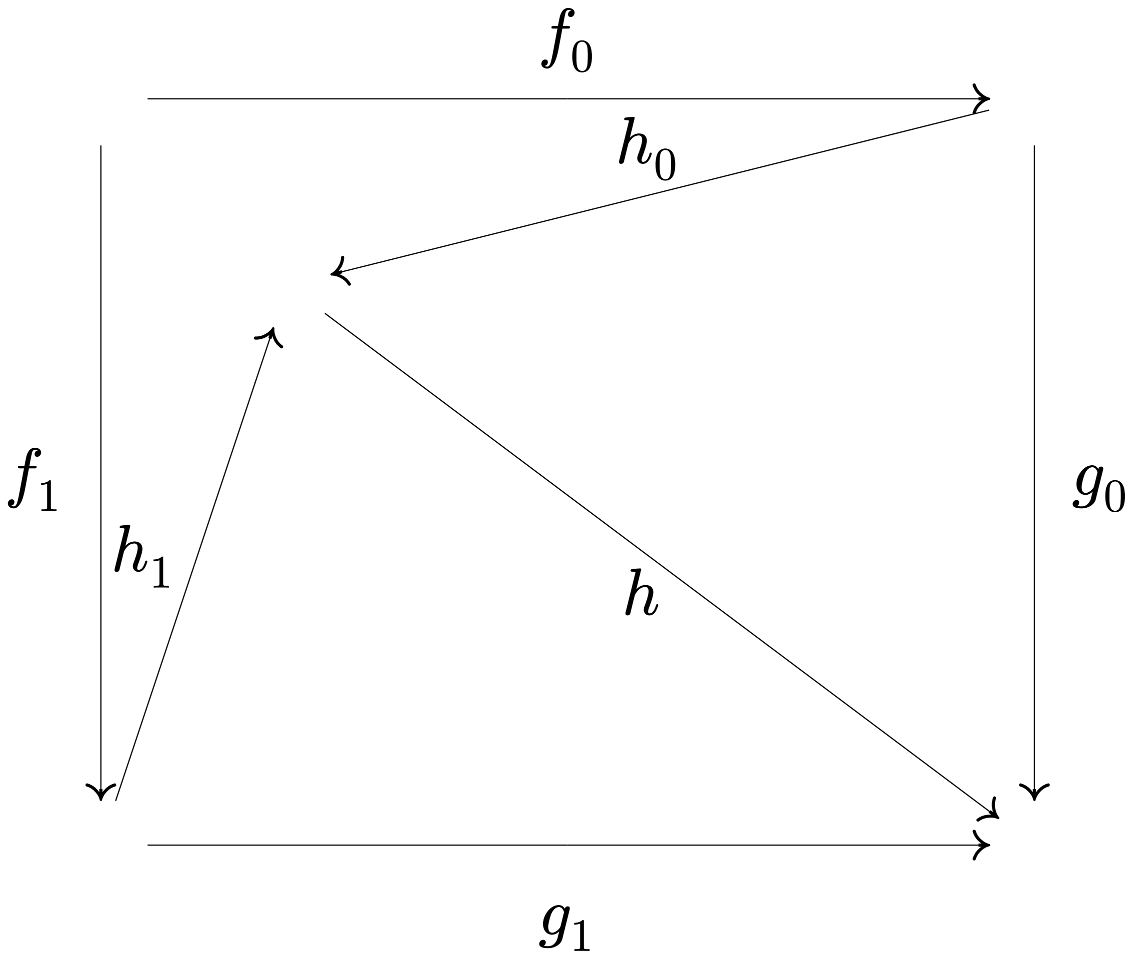
Definition Relative pushout

Given a commuting square $g_0 \cdot f_0 = g_1 \cdot f_1$ a relative pushout is a triple (h_0, h_1, h) satisfying:

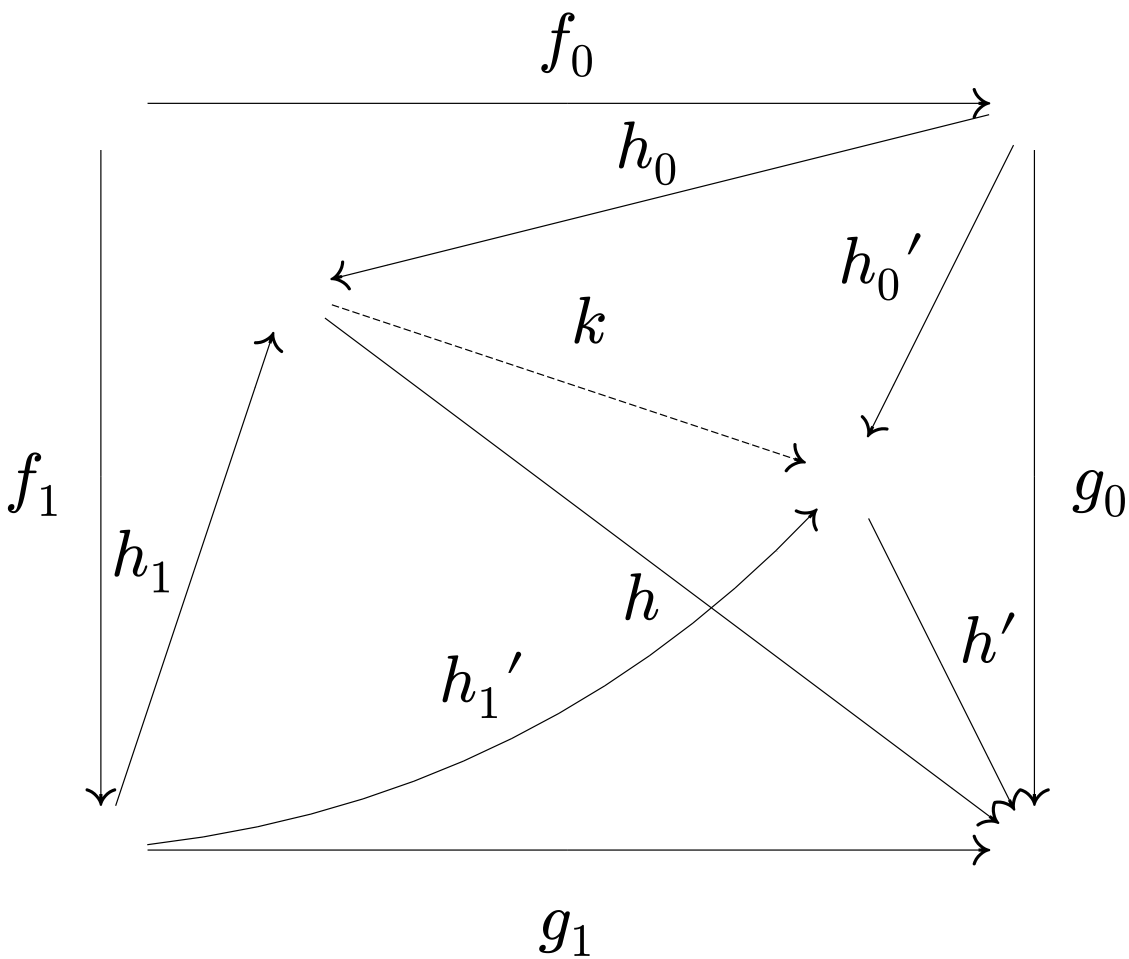
- Commutation: $h_0 \cdot f_0 = h_1 \cdot f_1$ and $\forall i \in \{0, 1\} h \cdot h_i = g_i$
- Universality: for any other $({h_0}', {h_1}', h')$ satisfying the universality constraint $\exists! k$ such that $h' \cdot k = h$ and $\forall i \in \{0, 1\}$ ${h'}_i \cdot k = h_i$



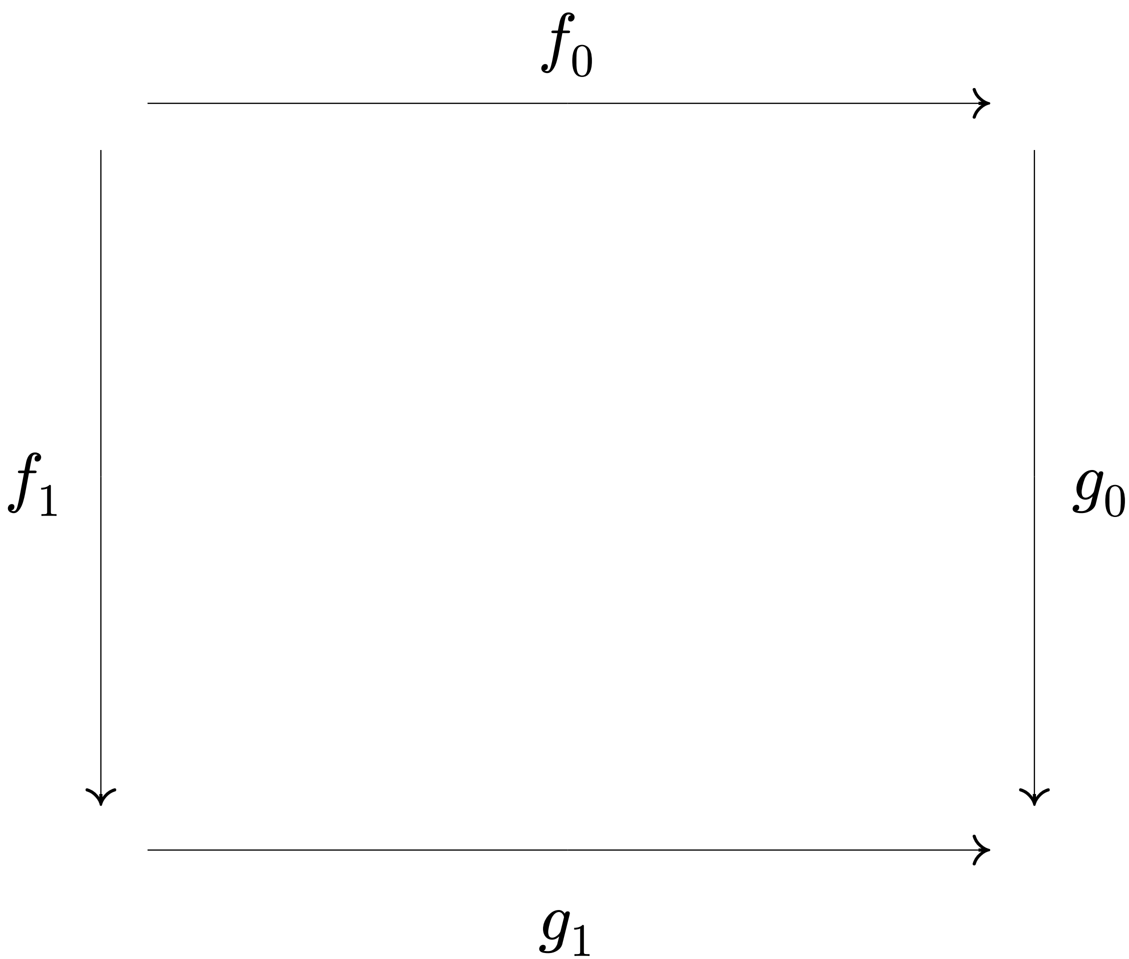
- Commuting Square



- Commuting Square
- Commutation of the RPO

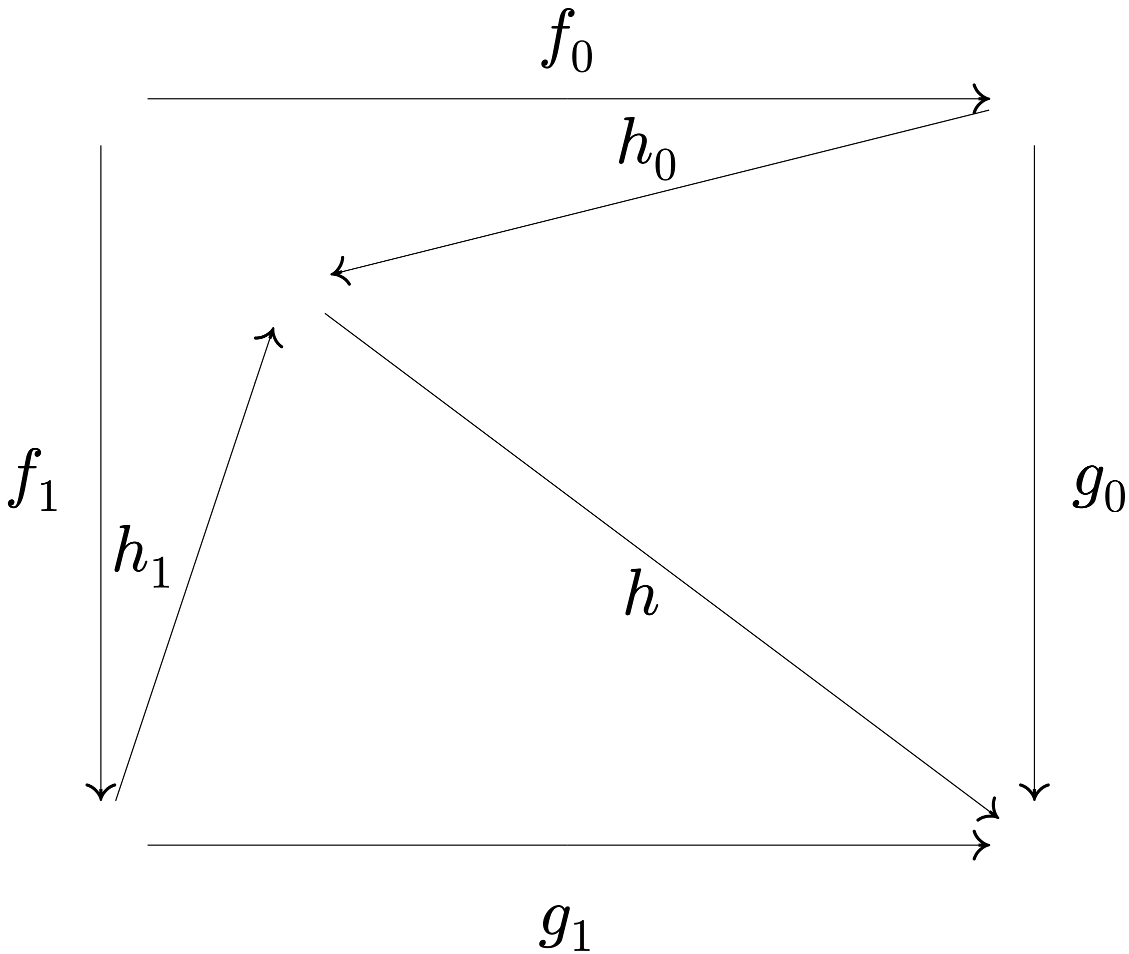


- Commuting Square
- Commutation of the RPO
- Universality of the RPO



Definition IPO

A commuting square $g_0 \cdot f_0 = g_1 \cdot f_1$ is an IPO if (g_0, g_1, id) is an RPO.



Definition IPO from RPO
if (h, h_1, h_2) is an RPO then the commuting square $h_0 \cdot f_0 = h_1 \cdot f_1$ is an IPO.

Transition

Definition Transition

$a \xrightarrow{F} a' \Leftrightarrow \exists (l, r) \in \text{React } \exists D \in \mathbb{D}$ such that $F \cdot a = D \cdot l$ is an IPO and $a' = D[r]$

$$\begin{array}{ccc} & \overset{a}{\longrightarrow} & \\ l \downarrow & & \downarrow F \\ & \underset{D}{\longrightarrow} & \end{array}$$

We are fixing the older definition of transition keeping only the "minimal" labels thanks to the IPO condition.

Bisimulation

Definition Simulation

$S \subseteq \bigcup_m C(0,m)^2$ is a simulation $\Leftrightarrow$ if $\forall (a,b) \in S$ if $a \xrightarrow{F} a'$ then $\exists b'$ such that $b \xrightarrow{F} b'$ and $(a',b') \in S$.

Definition Bisimulation

S is a bisimulation $\Leftrightarrow$ S and S^{-1} are simulations.

Definition Bisimilarity

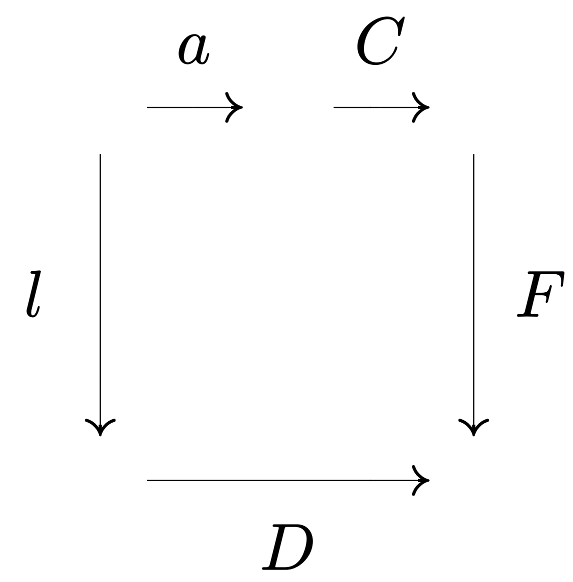
$\sim$ is the largest bisimulation.

Congruence

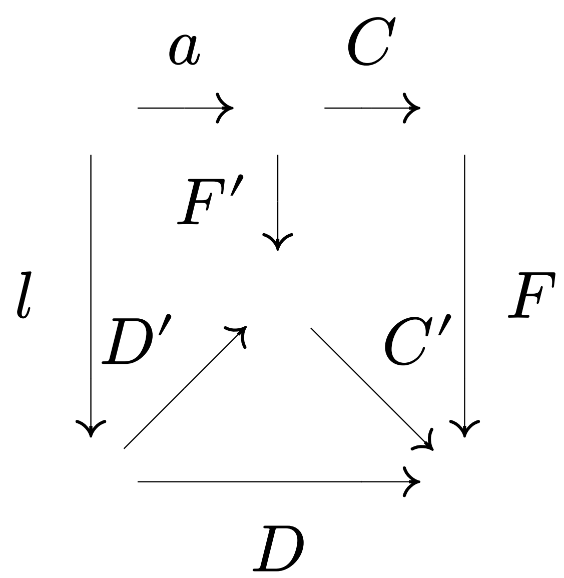
Definition redex-rpo
$\mathbb{C}$ has all redex-rpo if $\forall (l, r) \in$ React, a, F, D such that the square $F \cdot a = D \cdot l$ commutes, the square has an rpo.

Proposition if $\mathbb{C}$ has all redex-RPO $a \sim b \Rightarrow \forall C \quad C[a] \sim C[b]$
We prove that $\{(C[a], C[b]) \mid a \sim b\}$ is a bisimulation.

if $C[a] \xrightarrow{F} a'$ we have the following IPO



Since ℂ has all redex-RPO



Since we can get IPO's from RPO's we know that the first diagram is an IPO, and from IPO-pasting we know that also the second one is an IPO.

$$\begin{array}{ccc} & \xrightarrow{a} & \\ l \downarrow & & \downarrow F' \\ & \xrightarrow[D']{} & \end{array} \qquad\qquad \begin{array}{ccc} & \xrightarrow{C} & \\ F' \downarrow & & \downarrow F \\ & \xrightarrow[C']{} & \end{array}$$

Since $a \sim b$ we get the commuting diagram

$$\begin{array}{ccc} & \overset{b}{\longrightarrow} & \\ l' \downarrow & & \downarrow F' \\ & \underset{E'}{\longrightarrow} & \end{array}$$

By IPO pasting on it we get the IPO

$$\begin{array}{ccccc} & \xrightarrow{b} & & \xrightarrow{C} & \\ l' \downarrow & & & & \downarrow F \\ & \xrightarrow[E']{} & & \xrightarrow[C']{} & \end{array}$$

That implies $C[b] \xrightarrow{F} b'$ and $a' \sim b'$ because $a' = C'[D'[r]]$, $b' = C'[E'[r']]$ and from $a \sim b$ we know that $D'[r] \sim E'[r]$.

Usual definitions

We can recover τ-like and weak transitions:

Definition $\xrightarrow[2]{F}$

$$a \xrightarrow[2]{F} a' \Leftrightarrow \begin{cases} F[a] \longrightarrow a' & \text{if } F \text{ is an isomorphism} \\ a \xrightarrow{F} a' & \text{otherwise} \end{cases}$$

Definition $\xRightarrow{F}$

$$a \xRightarrow{F} a' \Leftrightarrow \begin{cases} F[a] \longrightarrow^{*} a' & \text{if } F \text{ is an isomorphism} \\ a \xrightarrow{F} \longrightarrow^{*} a' & \text{otherwise} \end{cases}$$

The bisimulations induced by these definition are all congruence.

Unnecessary labels when we introduce depth

The issue raises when we can nest complete copies of terms that can reduce by themselves we get labels that are unnecessary.

Consider a ractive system containing the rule $(\gamma(\alpha), \alpha')$ using our usual definition we would get the following reaction rule
$\alpha' \xrightarrow{\beta(\cdot,\gamma(\alpha))} \beta(\alpha', \alpha')$

We can fix the issue by considering multi hole contexts.

Definition Multi-hole reactive systems

A reactive system is a 4-tuple $((\mathbb{C}, \otimes, 0), Z, \mathbb{G}, \mathrm{React})$ where

- $(\mathbb{C}, \otimes, 0)$ is a strictly monodial category.
- $Z \subseteq |\mathbb{C}|$
- $\mathrm{React} \subseteq \bigcup_{m \in Z} \mathbb{C}(0, m)^2$ is the set of reaction rules.
- $\mathbb{D} \subseteq \mathbb{C}$ made of reactive contexts and composition reflecting and $a \otimes \mathrm{id}_m \in D \quad \forall a : 0 \to m'$

Arrows $0 \to m$ are agents and arrows $m \to m'$ are contexts that take m arguments and returns an m'-tuple of terms $(m, m' \in Z)$.

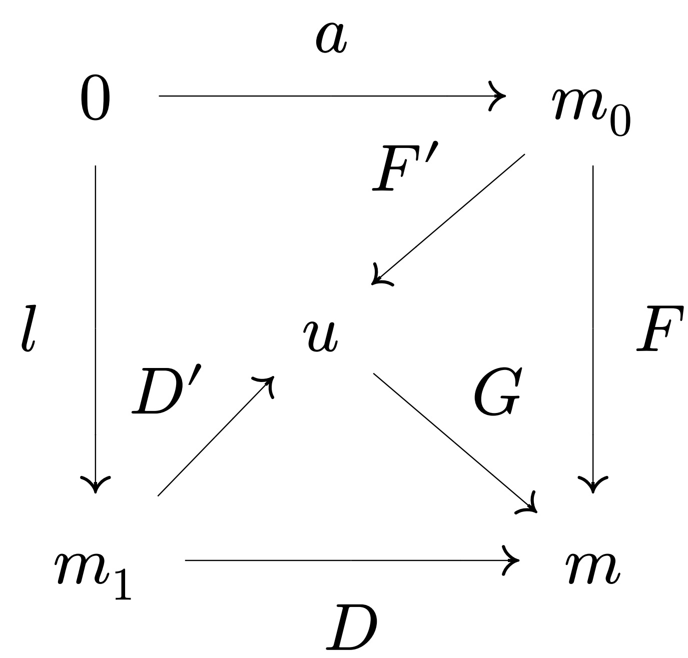
Definition Transition

$a \xrightarrow{F} a' \Leftrightarrow a, a'$ are agents, F is a context and $\exists (l, r) \in \text{React}\ \exists D \in \mathbb{D}$ such that $F \cdot a = D \cdot l$ is an IPO and $a' = D[r]$

$$
\begin{array}{ccc}
 & a & \\
 & \longrightarrow & \\
l \downarrow & & \downarrow F \\
 & \longrightarrow & \\
 & D &
\end{array}
$$

Definition Redex-RPO

$\mathbb{C}$ has all redex-RPO if $\forall (l, r) \in$ React and a agent, F, D contexts such that the square $F \cdot a = D \cdot l$ commutes, has an RPO such that either $u \in Z$ or $\exists k : u \to m_0 \otimes m_1$ isomorphism such that such that $k \cdot F' = \mathrm{id}_{m_0} \otimes l$ and $k \cdot D' = a \otimes \mathrm{id}_{m_1}$.


a
0
m0
F′
l
u
F
D′
G
m1
m
D

Definition Tensor IPO

$\mathbb{C}$ has all tensor IPO if the square $a_0 \cdot a_0 \otimes \mathrm{id}_{m_0} = a_1 \cdot a_1 \otimes \mathrm{id}_{m_1}$ is an IPO $\forall a_i : 0 \to m_i$ where $m_i \in Z$.

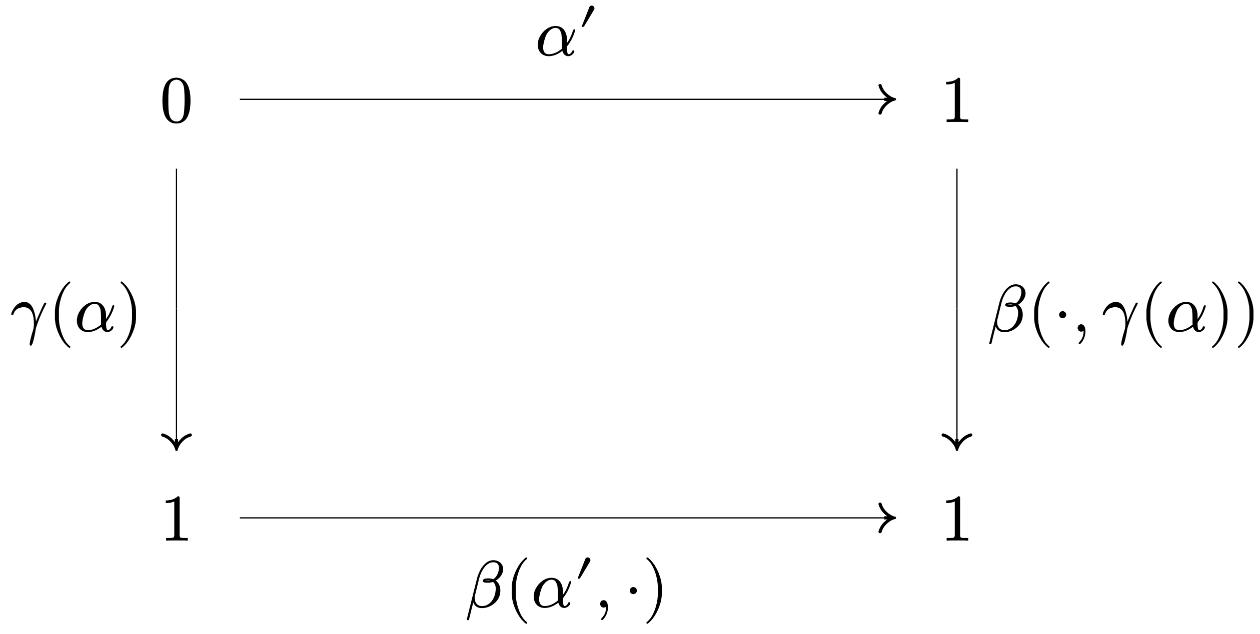
$$
\begin{array}{ccc}
0 & \xrightarrow{\;a_0\;} & m_0 \\
{\scriptstyle a_1}\downarrow & & \downarrow{\scriptstyle \mathrm{id}_{m_0} \otimes a_1} \\
m_1 & \xrightarrow[\;a_0 \otimes \mathrm{id}_{m_1}\;]{} & m_0 \otimes m_1
\end{array}
$$

Proposition if $\mathbb{C}$ has all redex-RPO and all tensor-IPO
$a \sim b \Rightarrow \forall C \quad C[a] \sim C[b]$

Example

Consider a system with a reaction rule: $(\gamma(\alpha), \alpha')$

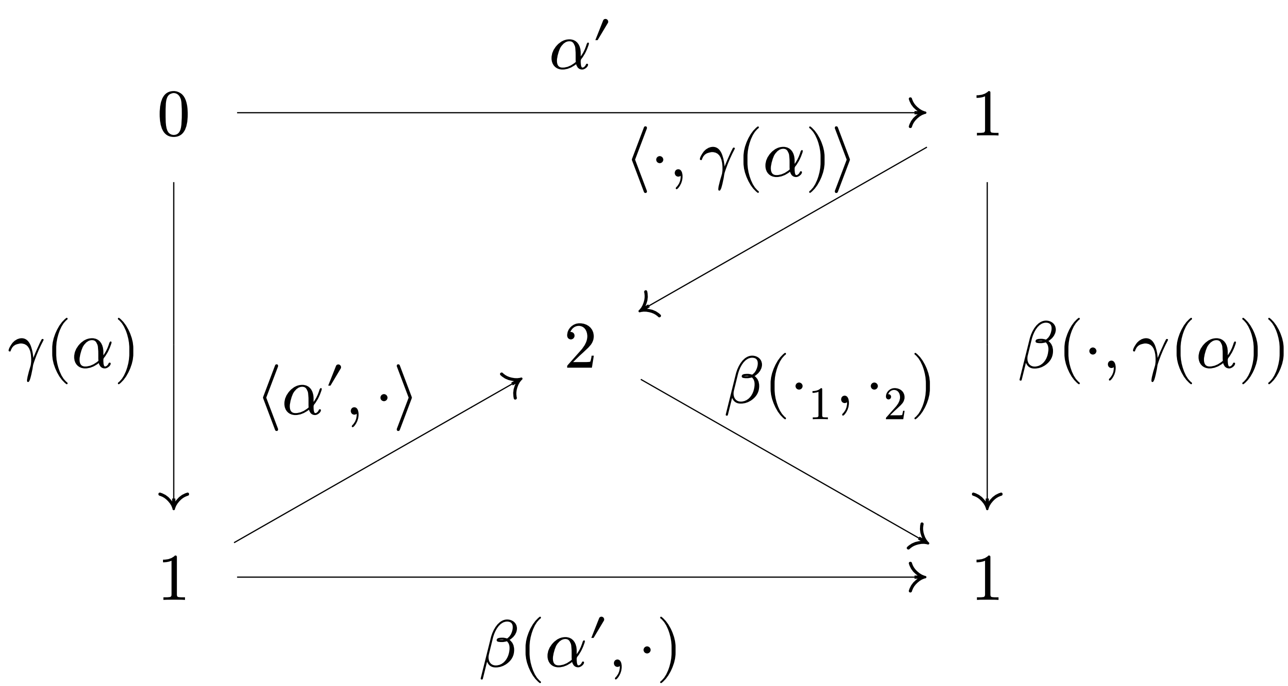
With one hole contexts:



Example

Consider a system with a reaction rule: $(\gamma(\alpha), \alpha')$

With multi hole contexts:



In CCS it doesn't work

For example the term $a.\,0|\bar{a}.\,0$ can perform the following transitions: $a.\,0|\bar{a}.\,0 \xrightarrow{\tau} 0$, $a.\,0|\bar{a}.\,0 \xrightarrow{a} a.\,0$ and $a.\,0|\bar{a}.\,0 \xrightarrow{\bar{a}} \bar{a}.\,0$ that should give us the following 3 IPOs:

$$
\begin{array}{ccc}
 & \xrightarrow{a.\,0|\bar{a}.\,0} & \\
a.\,0|\bar{a}.\,0 \downarrow & & \downarrow \cdot \\
 & \xrightarrow[\cdot]{} &
\end{array}
\qquad
\begin{array}{ccc}
 & \xrightarrow{a.\,0|\bar{a}.\,0} & \\
a.\,0|\bar{a}.\,0 \downarrow & & \downarrow \cdot\,|a \\
 & \xrightarrow[\cdot\,|a]{} &
\end{array}
\qquad
\begin{array}{ccc}
 & \xrightarrow{a.\,0|\bar{a}.\,0} & \\
a.\,0|\bar{a}.\,0 \downarrow & & \downarrow \cdot\,|\bar{a} \\
 & \xrightarrow[\cdot\,|\bar{a}]{} &
\end{array}
$$

Clearly we can "factorize" $\cdot\,|a$ and $\cdot\,|\bar{a}$ and obtain $\cdot$ hence the last 2 diagrams can't be IPOs.

Thanks for the attention!